Determining Liouvillian First Integrals for Dynamical Systems in the Plane

^{a,b}J. Avellar, ^{a}L.G.S. Duarte, ^{c} S.E.S. Duarte and ^{a}L.A.C.P. da Mota [1]

^{a} *Universidade do Estado do Rio de Janeiro,*
Instituto de Física, Depto. de Física Teórica,
20559-900 Rio de Janeiro – RJ, Brazil

^{b} *Fundação de Apoio à Escola Técnica,*
E.T.E. Juscelino Kubitschek,
21311-280 Rio de Janeiro – RJ, Brazil

^{c} *C.F.E.T ,*
20537-200 Rio de Janeiro – RJ, Brazil

Abstract

Here we present/implement an algorithm to find Liouvillian first integrals of dynamical systems in the plane. In [1], we have introduced the basis for the present implementation. The particular form of such systems allows reducing it to a single rational first order ordinary differential equation (rational first order ODE). We present a set of software routines in Maple 10 for solving rational first order ODEs. The package present commands permitting research incursions of some algebraic properties of the system that is being studied.

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[1]E-mails: javellar@dft.if.uerj.br, lduarte@dft.if.uerj.br, sduarte@dft.if.uerj.br and damota@dft.if.uerj.br

PROGRAM SUMMARY

Title of the software package: Lsolver.

Catalogue number: (supplied by Elsevier)

Software obtainable from: CPC Program Library, Queen's University of Belfast, N. Ireland.

Licensing provisions: none

Operating systems under which the program has been tested: Windows ME, Windows XP.

Programming languages used: Maple 10

Memory required to execute with typical data: 128 Megabytes.

No. of lines in distributed program, including On-Line Help, etc.: 900.

Keywords: Liouvillian functions, first integrals, dynamical systems in the plane, first order ordinary differential equations, computer algebra, Prelle-Singer (PS).

Nature of mathematical problem
Solution of rational first order ordinary differential equations.

Methods of solution
The method of solution is based on a Darboux/PS type approach.

Restrictions concerning the complexity of the problem
If the integrating factor for the rational first order ODE under consideration presents Darboux Polynomials of high degree (> 3) in the dependent and independent variables, the package may spend an unpractical amount of time to obtain the solution.

Typical running time
This depends strongly on the ODE, but usually under 2 seconds.

Unusual features of the program
Our implementation not only search for Liouvillian conserved quantities, but can also be used as a research tool that allows the user to follow all the steps of the Darboux procedure (for example, the algebraic invariants curves and associated cofactors can be calculated). In addition, since our package is based in recent theoretical developments [1], it can successfully solve a class of rational first order ODEs that were not solved by some of the best-known ODE solvers available.

LONG WRITE-UP

1 Introduction

The problem of searching for conserved quantities (first integrals) on physical systems is not new. In the last thirty years there has been a great increasing of interest in methods that follow the lines drawn by Darboux [2] in the second half of 19th century combined with topics that comes from differential algebra (developed by Ritt [3] and Kolchin[4] in the middle of the 20th century).

Particularly, the polynomial systems on the plane,

$$
\begin{aligned}
\frac{dx}{dt} &= N(x,y) \\
\frac{dy}{dt} &= M(x,y)
\end{aligned}
\tag{1}
$$

where M and N are polynomials in (x,y), have received a great deal of attention. We can then notice that there is an equivalence in finding first integrals of this system with the solving of a rational first order differential equation of the form:

$$
y' = \frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}.
\tag{2}
$$

If $I(x,y)$ is a first integral of the system (1) then $I(x,y) = K$, where K is a constant, is a general solution of the first order differential equation (2).

A big step forward in constructing an algorithm for solving first order ODEs analytically was taken in a seminal paper by Prelle and Singer (PS) [5] on autonomous systems of ODEs. Prelle and Singer's problem is equivalent to asking when a rational first order ODE of the form (2) has an elementary solution (a solution which can be written in terms of a combination of polynomials, logarithms, exponentials and radicals). Prelle and Singer were not exactly able to construct an algorithm for solving their problem, since they were not able to define a degree bound for the polynomials which might enter into the solution. Though important from a theoretical point of view, any computer implementation of the PS method will have a practical degree bound imposed by the processing speed and/or memory needed to handle the ever-more complex equations. With this in mind it is possible to say that Prelle and Singer's original method is almost an algorithm, awaiting a theoretical degree bound to turn it algorithmic. The approach of Prelle and Singer [5] was so interesting that it has motivated many extensions to its main ideas [6, 7, 8, 9, 10, 11]

The purpose of this paper is to describe an implementation, in Maple, of a Darboux-type procedure (presented on [1]) that extends the applicability of the PS-approach to deal with rational first order ODEs (2) with Liouvillian [2] solutions.

The paper is organized as follows: in section 2, we present a short theoretical introduction to the PS approach; in the following section, we introduce the main ideas of our extension [1]; in section 4, we present examples of the application of our method; in section 5, we present the *Lsolver* package; section 6 talk about some interesting features and performance of our package.

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[2]For a formal definition of Elementary and Liouvillian functions, please see [12].

2 The State of the art

Despite its usefulness in solving rational first order ODEs, the Prelle-Singer procedure is not very well known outside mathematical circles, and so we present a brief overview of the main ideas of the original PS-procedure and some of its extensions.

2.1 The Prelle-Singer Method

Consider the class of rational first order ODEs which can be written as

$$y' = \frac{dy}{dx} = \frac{M(x,y)}{N(x,y)} \tag{3}$$

where $M(x,y)$ and $N(x,y)$ are polynomials with coefficients in the complex field C.

In [5], Prelle and Singer proved that, if an elementary first integral of (3) exists, it is possible to find an algebraic integrating factor R such that R^n is a rational function of (x,y) for some integer n. If one knows R, then the ODE can be solved by quadrature.

We have that:

$$\frac{\partial(RN)}{\partial x} + \frac{\partial(RM)}{\partial y} = 0. \tag{4}$$

From (4) we see that

$$N\frac{\partial R}{\partial x} + R\frac{\partial N}{\partial x} + M\frac{\partial R}{\partial y} + R\frac{\partial M}{\partial y} = 0. \tag{5}$$

Thus

$$\frac{D[R]}{R} = -\left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right), \tag{6}$$

where

$$D \equiv N\frac{\partial}{\partial x} + M\frac{\partial}{\partial y} \tag{7}$$

and

$$R = \prod_i f_i^{n_i} \tag{8}$$

where f_i are irreducible polynomials and n_i are non-zero rational numbers. From (7) and (8), we have

$$\frac{D[R]}{R} = \frac{D[\prod_i f_i^{n_i}]}{\prod_i f_k^{n_k}} = \frac{\sum_i f_i^{n_i-1} n_i D[f_i] \prod_{j\neq i} f_j^{n_j}}{\prod_k f_k^{n_k}}$$

$$= \sum_i \frac{f_i^{n_i-1} n_i D[f_i]}{f_i^{n_i}} = \sum_i \frac{n_i D[f_i]}{f_i}. \tag{9}$$

From (6), plus the fact that M and N are polynomials, we conclude that $D[R]/R$ is a polynomial. Therefore, from (9), we see that $f_i|D[f_i]$ (i.e., f_i is a divisor of $D[f_i]$).

We now have a criterion for choosing the possible f_i (build all the possible divisors of $D[f_i]$) and, by using (6) and (9), we have

$$\sum_i \frac{n_i D[f_i]}{f_i} = -\left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right). \tag{10}$$

If we manage to solve (10) and thereby find n_i, we know the integrating factor for the ODE and the problem is reduced to a quadrature.

2.2 Extensions to the Prelle and Singer method

The original PS-procedure deals with rational first order ODEs with elementary solutions. Extensions were produced, both on the theoretical basis and on producing new algorithms, in order to allow tackling more general ODEs.

Using the results presented on [13], it is possible to shown [9, 7, 8] that, if the solution for a rational first order ODE of form (3) can be written in terms of Liouvillian functions, then it presents an integrating factor is of the form:

$$R = e^{r_0(x,y)} \prod_{i=1}^{n} v_i(x,y)^{c_i} = e^{P(x,y)/Q(x,y)} S(x,y), \qquad (11)$$

where $r_0 = P/Q$ (P and Q are polynomials in (x,y)) is a rational function of (x,y), $D[r_0]$ is a polynomial, $S = \prod_j v_i^{c_i}$, the v_i are Darboux polynomials (eigenpolynomials) of the D operator and c_i are constants.

Note that this result is the analogous of the previously mentioned result due to Prelle-Singer but for a more general case, namely, now we are dealing with Liouvillian solutions (please note that the elementary solutions, dealt with by Prelle and Singer, are also Liouvillian but are a restriction to the general Liouvillian case).

In [7, 8], we introduced a method that, using the knowledge embodied by (11), solves the ODE for the cases where r_0 is either $f(x)$, $g(y)$ or $f(x) + g(y)$, where f and g are rational functions.

In the next section, we will go a further step by introducing a new method that allows for the general case for the rational function r_0.

3 Introducing the new Method

In what follows, we will refine the result (11) by uncovering information about the structure of Q ($r_0 = P/Q$). This further knowledge will then be converted into something that will allow us to build a solving method.

Using (11) into $D[R]/R = -(\partial_x N + \partial_y M)$, we get:

$$D\left[\frac{P}{Q}\right] + \frac{D[S]}{S} = -\left(\partial_y M + \partial_x N\right), \qquad (12)$$

leading to

$$\frac{Q\,D[P] - P\,D[Q]}{Q^2} + \sum_j c_j \frac{D[v_j]}{v_j} = -\left(\partial_y M + \partial_x N\right). \qquad (13)$$

Remembering that the v_j are Darboux polynomials of the D operator, we can write $D[v_j] = \lambda_j\, v_j$, where the λ_j are polynomials in (x,y) (called cofactors) associated with the v_j. Thus, we can write (13), multiplying both sides by Q^2, as:

$$Q\,D[P] - P\,D[Q] + Q^2 \sum_j c_j\, \lambda_j = -Q^2\left(\partial_y M + \partial_x N\right). \qquad (14)$$

The above equation would be the analogous (in our method) to equation (6) (in the PS method). We call the attention to the fact that we are trying to deal with a much wider class of ODEs. This is embodied by the presence of the term $e^{P/Q}$ on the integrating factor. So far, we only know that P and Q are polynomials thus, it is easy to see that, attempting to solve (14), actually means solving a third degree

equation for the unknowns: c_j and the coefficients defining the polynomials P and Q. So, our case here seems much poorer than the PS original one as a practical approach to solving rational first order ODEs (albeit dealing with more general ones).

Can we do something to improve that situation? Actually, we will show that we can by presenting a refinement to our knowledge about the general structure for the integrating factor for ODEs of the form (3).

Theorem: *Let the exponent r_0 be expressed as $P(x,y)/Q(x,y)$, where P and Q are polynomials in (x,y), with no common factor. Then we have that $Q|D[Q]$ (i.e., $D[Q]/Q$ is a polynomial in (x,y)).*

Proof: Since $D[r_0]$ is polynomial (see [8]), we can write it as:

$$D[r_0] = D\left[\frac{P}{Q}\right] = \frac{Q\,D[P] - P\,D[Q]}{Q^2} = \Pi \tag{15}$$

where Π is polynomial in (x,y). Multiplying (15) by Q, one obtains:

$$D[P] - \frac{P\,D[Q]}{Q} = \Pi Q. \tag{16}$$

Since D is a linear differential operator, with polynomial coefficients, and P is polynomial, $D[P]$ is also polynomial. Therefore, since $\Pi\,Q$ is also polynomial, we may conclude that $\frac{P\,D[Q]}{Q}$ is polynomial either. Since, by hypothesis, P and Q have no common factor, we can infer that $Q|D[Q]$ (i.e., $D[Q]/Q$ is a polynomial), as we wanted to demonstrate [3].

This result, in turn, leads to the following corollary:

Corollary: *We can write Q as $\prod_{i=1} q_i(x,y)^{m_i}$, where the q_i are irreducible independent Darboux polynomials of the D operator and the m_i are positive integers.*

Proof: Q is a polynomial so it can be written as $\prod_{i=1} q_i(x,y)^{m_i}$, where the q_i are independent irreducible polynomials and the m_i's are positive integers. So, we can write:

$$\frac{D[Q]}{Q} = \sum_i m_i \frac{D[q_i]}{q_i}, \tag{17}$$

from the theorem just proved, we know that the left-hand side of (17) is polynomial. So the right-hand side is also polynomial implying that $q_i|D[q_i]$ (i.e., $D[q_i]/q_i$ is a polynomial), as we wanted to show.

Using these results, we can write $Q = \prod_i \left(v_{qi}\right)^{m_i}$, where the v_{qi} are independent irreducible Darboux polynomials of the D operator, and so:

$$\frac{D[Q]}{Q} = \sum_i m_i \frac{D[v_{qi}]}{v_{qi}} = \sum_i m_i\,\lambda_{qi}, \tag{18}$$

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[3]An analogous result was demonstrated (by other means) on [10]

where the λ_{qi} are the cofactors associated with the v_{qi}. Using this into (14) and re-arranging we get:

$$D[P] - P \sum_i m_i \lambda_{qi} = - \prod_i (v_{qi})^{m_i} \left(\sum_j c_j \lambda_j + \partial_y M + \partial_x N \right). \qquad (19)$$

This equation will prove to be very important to our method. So, let us now do some analyzing of its structure: What is the advantage of (19) in comparison with (14)?

In words, now we know the structure of Q, i.e., the building blocks of Q. In (14), we had to set the degree for v_i (analogously, in the PS method, we had to set the degree for the f_i) and to set the degree for the P and Q polynomials (there is no analogous in the PS method). After that, as already pointed out, we are left to solve a third degree equation on the unknowns. Now, with (19), we have to do the same setting of degrees but, due to our improved knowledge about Q, after setting these degrees, we are left with a finite set of possibilities for the **integers** $\{m_i\}$. Thus converting solving (19) to solving a system of linear equations for the unknowns c_j and the coefficients defining P (one system of linear equations for each possible set of integers $\{m_i\}$).

So, after setting the degrees we mentioned, in essence, equation (19) is as manageable as equation (6) of the PS method. This is the basis of a new method to deal with rational first order ODEs that can be summarized as follows:

In equation (19) (the cornerstone of our method), we see terms involving the Darboux polynomials (and/or the associated cofactors) of the D operator. So, the first job at hand is to determine those. In order to do that we have to choose a degree (as in the PS method). For simplicity sake, let us start with the most simple possibility, namely, find the Darboux polynomials of degree 1 of the D operator and the associated cofactors. To be able to solve equation (19), we have to set the degree for the Q and P polynomials. Remembering that we know the structure for Q, setting the degree for Q allows us (see above) to find all possibilities for $\{m_i\}$. After this, we then try to solve (19). If we solve it, we would have found the integrating factor we were looking for. If we can not solve it, we have to change our settings and try again. The way we will change the settings[4] is a matter of choice and, in principle, it is difficult to see which one will be the best since this depends strongly on the ODE itself. The point is that, no matter which is that choice, the method is contained in the sense that, even if we have to use high values for the degree for Q and P, our task will be to solve linear equations (sure, it can be a great number of equations, but always linear ones).

Of course, this procedure can go on indefinitely, but we will be covering all the possibilities and we may hope that we will find a solution within our lifetime. On a brighter tone, all the examples we have come across are solved with degree for the Darboux polynomials of the D operator, Q and P that allow for a practical realization of the method.

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[4]For example, should we increase the degree of P without increasing the one for Q? Up to which value? When should we increase the degree for the Darboux polynomials of the D operator? etc.

4 The Inner Workings of the Method

In this section, we are going to present examples of application of the method. We will present the calculations in a long version in order for the reader to have a deeper understanding of the inner workings of the method.

First, in order to illustrate the method just presented, we are going to start with a simple ODE. This example was chosen due to its simplicity (so it is a good stating point for the introducing of the method) and for the fact that, even being simple, this ODE is not solved by the Maple powerful solver `dsolve`.

Example 1:

Consider the following ODE:

$$\frac{dy}{dx} = \frac{(x+1)\,y}{x - xy - y^2 + x^2} \tag{20}$$

For this equation, we have $M = (x+1)\,y$ and $N = x - xy - y^2 + x^2$ leading to:

$$D = N\partial_x + M\partial_y = (x - xy - y^2 + x^2)\partial_x + (x+1)\,y\partial_y \tag{21}$$

Up to degree 1, we have that the Darboux polynomials (with the associated cofactors) for this operator are:

- $v_1 = y, \quad \lambda_1 = x + 1,$

- $v_2 = x + y, \quad \lambda_2 = 1 + x - y.$

Then we have to choose the degree for the polynomial Q. Starting with degree 1, since v_1 and v_2 are of degree 1, the only possible values for m_i are $\{m_1 = 1, m_2 = 0\}$ and $\{m_1 = 0, m_2 = 1\}$. Starting with degree 1 for P, we have $P = a_1 + a_2\,x + a_3\,y$ and equation (19) leads to:

$$
\begin{aligned}
y^{m1}\,(x+y)^{m2}\,(n1\,(x+1) + n2\,(1+x-y) + 3\,x + 2 - y) + \\
\left(x - xy - y^2 + x^2\right) a2 + (x+1)\,y\,a3 - \\
(a1 + a2\,x + a3\,y)\,(m1\,(x+1) + m2\,(1+x-y)) = 0.
\end{aligned}
\tag{22}
$$

As we shall see, with the values $m_1 = 1, m_2 = 0$ it is possible to find a solution. Substituting $\{m_1 = 1, m_2 = 0\}$ into (22) we get:

$$
\begin{aligned}
y\,(n1\,(x+1) + n2\,(1+x-y) + 3\,x + 2 - y) + \left(x - xy - y^2 + x^2\right) a2 + \\
(x+1)\,y\,a3 - (a1 + a2\,x + a3\,y)\,(x+1) = 0.
\end{aligned}
\tag{23}
$$

In order to solve the above equation, the coefficients for different powers of (x, y) have to be zero. Thus leading to the following system of linear equations:

$$
\begin{aligned}
n1 + n2 + 2 &= 0 \\
-n2 - a2 - 1 &= 0 \\
-a1 &= 0 \\
n1 + n2 + 3 - a2 &= 0
\end{aligned}
\tag{24}
$$

Leading to the solution for the coefficients:

$$a1 = 0, \; a3 = a3, \; n1 = 0, \; n2 = -2, \; a2 = 1 \tag{25}$$

So, the integrating factor for this ODE becomes (choosing $a_3 = 0$):

$$R = \frac{e^{x/y}}{(x+y)^2} \tag{26}$$

Then the solution is:

$$C = \frac{y\,(-1+y)\,e^{x/y}}{(x+y)} - e^{-1}Ei(1, -\frac{x+y}{y}) \tag{27}$$

We are going to present now an example extracted from the book by Kamke [14]. This choice for the second example was made aiming to show that, even for involved cases, our method is contained and manageable.

Example 2:

Consider the following ODE (**I.169** from the book by Kamke):

$$(ax + b)^2 \frac{dy}{dx} + (ax + b)\, y^3 + cy^2 \tag{28}$$

For this equation, we have $M = -\left((ax+b)y^3 + cy^2\right)$ and $N = (ax+b)^2$ leading to:

$$D = N\partial_x + M\partial_y = (ax+b)^2\,\partial_x - \left((ax+b)y^3 + cy^2\right)\partial_y \tag{29}$$

For this equation, up to degree 1, we have that the Darboux polynomials (with the associated cofactors) for this operator are:

- $v_1 = y, \quad \lambda_1 = -y\,c - b\,y^2 - a\,x\,y^2,$
- $v_2 = (ax + b)/a, \quad \lambda_2 = a\,b + a^2\,x.$

Then we have to choose the degree for the polynomial Q. For this particular example, we will see that we need to use degree 4 for both Q and P. Since v_1 and v_2 are of degree 1, the possible values for m_i are $\{m_1 = 4, m_2 = 0\}$, $\{m_1 = 3, m_2 = 1\}$, $\{m_1 = 2, m_2 = 2\}$, $\{m_1 = 1, m_2 = 3\}$ and $\{m_1 = 0, m_2 = 4\}$. Letting $P = a_1\,y^4 + a_2\,x^2\,y + a_3\,y^2 + a_4\,x\,y^2 + a_5\,x\,y + a_6\,x^4 + a_7\,x^3 + a_8\,x^2 + a_9\,y^3 + a_{10}\,x^3\,y + a_{11}\,x\,y^3 + a_{12}\,x + a_13\,x^2\,y^2 + a_{14}\,y + a_{15}$, equation (19) becomes:

$$y^{m1}\left(\frac{ax+b}{a}\right)^{m2}\left(n1\left(-yc - by^2 - axy^2\right) + n2\left(ab + a^2x\right)\right.$$

$$-2\,y\,(axy + by + c) - y^2\,(ax+b) + 2\,a^2x + 2\,ab\Big)$$

$$+\left(a^2x^2 + 2\,axb + b^2\right)\left(2\,a2\,xy + a4\,y^2 + a5\,y+\right.$$

$$4\,a6\,x^3 + 3\,a7\,x^2 + 2\,a8\,x + 3\,a10\,x^2y + a11\,y^3 + a12 + 2\,a13\,xy^2\Big)$$

$$-y^2\,(axy + by + c)\left(4\,a1\,y^3 + a2\,x^2 + 2\,a3\,y + 2\,a4\,xy + a5\,x + 3\,a9\,y^2\right.$$

$$+a10\,x^3 + 3\,a11\,xy^2 + 2\,a13\,x^2y + a14\Big) - \left(a1\,y^4 + a2\,x^2y + a3\,y^2 + a4\,xy^2\right.$$

$$+a5\,xy + a6\,x^4 + a7\,x^3 + a8\,x^2 + a9\,y^3 + a10\,x^3y + a11\,xy^3 + a12\,x$$

$$+a13\,x^2y^2 + a14\,y + a15\Big)\left(m1\left(-yc - by^2 - axy^2\right) + m2\left(ab + a^2x\right)\right) = 0 \tag{30}$$

As we shall see, with the values $m_1 = 2, m_2 = 2$, it is possible to find a solution. Substituting those values for m_1 and m_2 into (30) we get:

$$\left\{ 4\,b^2\,a6 + 6\,aba7 + 2\,a^2\,a8 = 0, 3\,b^2\,a7 + a^2\,a12 + 4\,aba8 = 0, \right.$$
$$2\,aba5 + a12\,c + n2\,a^2 + 2\,b^2\,a2 + 2\,a^2 = 0, -3\,ca1 - 2\,ba9 = 0,$$
$$b^2\,a12 = 0, -n1\,c + a15\,b + b^2\,a4 - 2\,c = 0, 2\,a^2\,a2 + a7\,c + 6\,aba10 = 0,$$
$$8\,aba6 + 3\,a^2\,a7 = 0, 2\,b^2\,a8 + 2\,aba12 = 0, 4\,a^2\,a6 = 0,$$
$$-3\,b - n1\,b + b^2\,a11 - ca3 = 0, a^2\,a5 + 3\,b^2\,a10 + a8\,c + 4\,aba2 = 0,$$
$$ba12 + a15\,a + 2\,b^2\,a13 + 2\,aba4 = 0, -n1\,a - 3\,a - ca4 + 2\,aba11 = 0,$$
$$ba8 + a^2\,a4 + aa12 + 4\,aba13 = 0, n2\,ab + 2\,ab + b^2\,a5 + a15\,c = 0,$$
$$a^2\,a11 - ca13 = 0, -ba3 - 2\,ca9 = 0, a6\,a = 0, 3\,a^2\,a10 + a6\,c = 0,$$
$$-2\,ba11 - 2\,aa9 = 0, -3\,aa1 = 0, -3\,ba1 = 0, -aa13 = 0, -2\,aa11 = 0,$$
$$-ba13 - aa4 = 0, -2\,ca11 - aa3 - ba4 = 0,$$
$$\left. aa8 + ba7 + 2\,a^2\,a13 = 0, aa7 + ba6 = 0 \right\} \quad (31)$$

Leading to the solution for the coefficients:

$$\left\{ a6 = 0, a11 = 0, a10 = 0, a1 = 0, n2 = -1, a2 = 0, a7 = 0, a9 = 0, \right.$$
$$a3 = -1/2\,\frac{c^2 - 2\,ab^2\,a13}{a^3}, n1 = -3, a14 = -\frac{cb}{a^2},$$
$$a15 = -1/2\,\frac{b^2}{a}, a5 = -\frac{c}{a}, a12 = -b, a4 = 2$$
$$\left. \frac{ba13}{a}, a13 = a13, a8 = -a/2 \right\} \quad (32)$$

So, the integrating factor for this ODE becomes (choosing $a_{13} = 0$):

$$R = \frac{e^{-\frac{\left(yc + a^2 x + ab\right)^2}{2\,ay^2(ax+b)^2}}}{y^3\,(ax+b)} \quad (33)$$

Then the solution is:

$$C = \int \frac{(axy + by + c)\,e^{\frac{-1}{2a}\left(\frac{c}{ax+b} + \frac{a}{y}\right)^2}}{y\,(ax+b)}dx \quad (34)$$

From the example above, we can see that, even in cases where Q and P have high degrees, our method convert the problem into a simple linear algebraic system. More specifically, in this example, the equations (31). Although these equations look frightening, they are solvable even by hand (if one is brave enough). In this computational implementation, the problem is trivially solved.

Let us then introduce our package implementing the above explained ideas and making the algorithm available for easy usage.

5 The *Lsolver* package

Summary

A brief review of the commands of the package is as follows:[5]

- `Lsolve` solves ODEs, using our Darboux-type approach. It deals only with rational first order ODEs.
- `IntFact` returns an integrating factor for the rational first order ODE.
- `LDop` constructs the D operator associated to the rational first order ODE.
- `Darboux` returns the DPs (Darboux polynomials) and cofactors associated with the D operator.

Description

A complete description of the *Lsolver* package's commands can also be found in the on-line help.

5.1 Command name: `Lsolve`

Feature: This command applies our Darboux-type procedure to solve ODEs.

Calling sequence[6]:

```
> Lsolve(ODE,extra_args);
```

Parameters:

`ODE`	- a first order ordinary differential equation.
`extra_args`	– `Deg = <bound>` - a positive integer equal to the maximum degree of the darboux polynomials. The default value is 1.
	– `Deg_Q = <bound_Q>` - a positive integer equal to the maximum degree of the Q polynomial; The default value is 5.
	– `Deg_P = <bound_P>` - a positive integer equal to the maximum degree of the P polynomial; The default value is 5.

Synopsis:

The `Lsolve` command is a part of the *Lsolver* package, which is designed to solve rational first order ODEs.

The arguments

The first argument of `Lsolve` is the ODE. The extra arguments are all positive integers that specify the polynomial degrees we are going to use in the search for

---

[5]This subsection and the next one may contain some information already presented in the previous sections; this is necessary to produce a self-contained description of the package.

[6]In what follows, the *input* can be recognized by the Maple prompt `>`.

the integrating factor. We would like to remind the reader that our Darboux-type procedure is a semi-decision one, in other words, if the integrating factor has not been found for these degrees, it does not exists at this level. But, it could as well exist for higher degree, we have to keep looking.

5.2   Command name: `IntFact`

Feature: Calculates an integrating factor for the ODE.

Calling sequence:

> `IntFact(ODE,extra_args);`

Parameters:

The parameters `ODE` and `extra_args` have the same meaning as explained above[7]. The command `IntFact` admits all `the extra_args` as the command `Lsolve`.

Synopsis:

   Actually this command embodies the heart of the package since, after finding the integrating factor, the solution is found via a quadrature.

5.3   Command name: `Darboux`

Feature: Calculates the Darboux polynomials, up to a certain degree, of the D operator related to the ODE in question. The default is to use degree=1 for the Darboux polynomials to be sought.

Calling sequence:

> `Darboux(ODE,extra_args);`

Parameters:

The command `Darboux` admits as `extra_args` only `Deg = <bound>`, where `<bound>` is a positive integer equal to the maximum degree of the darboux polynomials.

Synopsis:

   This command is very important to the method since it determines the building blocks for the integrating factor.

5.4   Command name: `LDop`

Feature: Returns the D operator (for the ODE) that is an essential ingredient in applying our procedure.

Calling sequence:

> `Dop(ODE);`

Synopsis:

   This command returns the D operator associated with the ODE.

---

[7]In what follows, these parameters will always have this meaning.

5.5 Example of the usage of the package commands

Let us use some rational first order ODEs to show the commands in action. Consider the following ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(1+x)}{x+x^2-y^2}. \tag{35}$$

Usually, people only want to find the solution for the ODE. So, that would simply require the input lines:

```
> eq := diff(y(x),x) = y(x)*(1+x)/(x+x^2-y(x)^2);
> Lsolve(eq);
```
                    The solution will be

$$-i\sqrt{\pi}\sqrt{2}erf\left(\frac{1/2\,i\sqrt{2}x}{y(x)}\right) + 2\,e^{1/2\,\frac{x^2}{(y(x))^2}}y(x) = \_C1 \tag{36}$$

Sometimes the user is interested on having a look at just the integrating factor, so he/she could type:

```
> IntFact(eq);
```
                  For the ODE in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(1+x)}{x+x^2-y^2} \tag{37}$$

              the integrating factor will be

$$\frac{\exp\left(\frac{x^2}{2y^2}\right)}{y^2} \tag{38}$$

Please notice that the program informs the user the format in which it is considering the ODE to be in, that is important since it may be the case where the input ODE gets transformed by the simplification routines and is regarded in a different shape (of course it is still the same ODE) and that will affect the integrating factor.

If one wants to find the Darboux polynomials (up to a certain degree), one can use the following input:

```
> Darboux(eq);
```

   Below, you can find a list containing two lists:
   [[Darboux polynomials],[co-factors]]

$$[[y],[1+x]] \tag{39}$$

Please remember that the default value for the degree of the Darboux polynomials is 1.

Finally, for this example, let us calculate the D operator:

```
> LDop(eq):
```

$$w \to (x + x^2 - y^2)(\frac{d}{dx}\ w) + y(1+x)(\frac{d}{dy}\ w) \qquad (40)$$

Let us deal with a more involved case. Consider the following ODE.

$$\frac{dy}{dx} = \frac{-1 + x + y + 3\,y^2}{2(2\,x + y + xy + y^2 - y^3)} \qquad (41)$$

Again, using the input lines just introduced above:

```
> eq := diff(y(x),x) = 1/2*(-1+x+y(x)+3*y(x)^2)/(2*x+y(x)+x*y(x)+
  y(x)^2-y(x)^3);
> Lsolve(eq);
```
<div align="center">We could not find an integrating factor</div>

What happened? The reason for the failure in solving the equation above could be due to the fact that there is no Darboux polynomials with degree equals to 1 and, since the default for the command assumes that value, we have to increase this value to check that possibility. This is done via the following input line:

```
> Lsolve(eq,Deg=2);
```

<div align="center">The solution will be</div>

$$2\,\sqrt{x + (y\,(x))^2}\,e^{1/4\,\frac{-1+4\,x+4\,y(x)}{x+(y(x))^2}} - 3\,\sqrt{\pi}e^1\,erf\left(1/2\,\frac{-1 + 2\,y\,(x)}{\sqrt{x + (y\,(x))^2}}\right) = \_C1 \qquad (42)$$

Let us have a look at the other commands:

```
> IntFact(eq,Deg=2);
```
<div align="center">For the ODE in the form</div>

$$\frac{dy}{dx} = \frac{-1 + x + y + 3\,y^2}{4\,x + 2\,y + 2\,yx + 2\,y^2 - 2\,y^3} \qquad (43)$$

<div align="center">the integrating factor will be</div>

$$e^{\frac{-1/4+x+y}{x+y^2}}\left(x + y^2\right)^{-3/2} \qquad (44)$$

The other two commands, for this case, produce:

```
> Darboux(eq,Deg=2);
```

```
  Below, you can find a list containing two lists:
  [[Darboux polynomials],[co-factors]]
```

$$[[x + y^2], [4 + 4\,y]] \qquad (45)$$

and

```
> LDop(eq):
```

$$w \to ((4x + 2y + 2yx + 2y^2 - 2y^3))(\frac{d}{dx}\ w) + (-1 + x + y + 3y^2)(\frac{d}{dy}\ w) \qquad (46)$$

<div align="center">14</div>

6 Performance

As previously mentioned, we have used rational ODEs to exemplify the usage of the package commands, we hope to have clarified that usage. Now we are going to further exemplify the package but with a different approach. We will not be concerned with the input/output of the commands. In order to analyze the contribution of the present implementation, we will comment on how the package extends the solving capabilities of the existing solvers and how it does that. We will divide that into two main groups and present examples of those:

6.1   *First order ODEs with Liouvillian solutions*

The first one is the most obvious. The present implementation tackles (semi-algorithmically) rational first order ODEs with Liouvillian solutions and those were not treated neither on the original PS-approach nor in its extensions. So, this implementation extends the applicability of these Darboux-type approaches to an wider range (not only on the Maple environment).

An example of a rational first order ODE belonging to this class is:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y^2(-2y + 1 - 2x + x^2 y)}{x^2(-2x + 1 - 2y + xy^2)} \tag{47}$$

Lsolver finds the following integrating factors for this ODE:

$$\frac{\exp\left(\frac{2x^2 + y^2}{x^2 y^2}\right)}{x^4 y^4}$$

which, in turn, lead to the following solution:

$$\int \frac{\exp\left(\frac{(x+y)^2}{x^2 y^2}\right)(-2y + 1 - 2x + x^2 y)}{x^2} dx = C$$

As previously mentioned, this equation would escape the standard PS procedure. Furthermore, this example (which is just an example of an infinite class) also elude the `dsolve` command.

We would like to point out that the Maple solver can deal with some members of this class, for instance the Abel's ODEs but not in the general case. Our previous implementation extending the Prelle-Singer method [15] also solve some ODEs but with heuristic approaches.

6.2   *First order ODEs with elementary solutions*

The second group talks about a bonus: The present implementation solves ODEs, with elementary solutions, that are missed by the powerful solvers on the Maple commercial algebraic package and in some of the (Maple) implementations of the PS-approach and its extensions already mentioned. We call that a bonus since the present package was designed to dwell in the previously uncharted region (at least semi-algorithmically) of the rational first order ODEs with Liouvillian solutions but it fared well elsewhere also, as we shall see. But, before we go on with the example,

let us answer the following: How can an implementation of the PS-approach [15] not solve ODEs with elementary solutions as we are claiming here?

Of course, theoretically, the PS-approach would eventually catch the integrating factor, given enough time and computational power. Here it is where the situation gets complicated (at least for ODEs like the one we are about to display). In practice, the Darboux polynomials can be of a too high a degree and that puts matters outside the nowadays computational capabilities (both on time and in computer memory). The attentive reader might ask: Why does this practical limitation not affect the extension here presented? The point is that the general form of the Integrating factor (11) can allow for a solution with lower Darboux polynomial degree to be found and that would turn the solving of the ODE feasible. Let us present then an example of these ODEs to clarify these ideas:

$$\frac{dy}{dx} = (-14\,x - 14\,y - 28\,x^3 + 14\,y^3 + 40\,x^4 - 58\,x^5 - 19\,x^2y + 30\,x^3y+$$

$$-23\,x^2y^2 + 26\,x^3y^2 + 14\,xy^3 + 21\,x^4y)\,/\,(x\,(7\,x^2 + 7\,x^3 + 7\,x+$$

$$+7\,y + 7\,xy + 7\,y^2 + 13\,x^2y + 7\,xy^2 + 13\,x^3y + 7\,x^4)) \tag{48}$$

Our program finds the following integrating factor for the above ODE:

$$R = e^{\frac{y-1}{x^2}}\,(x+1)^{-8} \tag{49}$$

which, in turn, lead to the following solution:

$$C = \frac{e^{\frac{y-1}{x^2}}x^3\,(7\,x + 7\,x^3 + x^4 - x^2y - 7\,x^2 + 7\,y^2 + 7\,y)}{(21\,x^5 + x^7 + 7\,x^6 + 35\,x^4 + 35\,x^3 + 21\,x^2 + 7\,x + 1)} \tag{50}$$

From (50) and (49), one can see that (48) admits an integrating factor $\overline{R}$ of the form:

$$\overline{R} = \frac{21\,x^5 + x^7 + 7\,x^6 + 35\,x^4 + 35\,x^3 + 21\,x^2 + 7\,x + 1}{(x+1)^8\,x^3\,(7\,x + 7\,x^3 + x^4 - x^2y - 7\,x^2 + 7\,y^2 + 7\,y)} \tag{51}$$

So, in principle, the PS-approach should find it. As we pointed out above, in practice this does not occur since (please see (51)) one would need to find Darboux polynomials up to degree 7. In our case, please look at (49), we needed to use Darboux polynomials of degree 1.

We remind the reader that the `dsolve` command contains many heuristic methods based on Lie theory for solving ordinary differential equations [16, 17]. But, because of their pattern-matching based nature, there will always be an infinity of cases (of which the equation above is an example), for which the symmetries can not be found.

6.3 Time Performance

In order to provide information regarding the computational time performance of the package in solving ODEs, we present the table 1 with the ODEs we have analysed in the paper. The reason for this choice is that, no matter how large the sample we

chose to display, it will never be exhaustive, there are an infinity of possibilities. So, we have decided to use the ODES to which the readers have more familiarity (since they have seen the equations being discussed) in order for them to better analyze which feature influences the most the time of solution. Only one ODE analyzed in the paper was left out of the table, equation (48). That is so, since it is too big for it to fit the table in a reasonable way. The time the program took to solve this ODE was $1.713\,s$ (please see the data related to the other ODEs on the table). As can be seen from the comparison of these data, the time the program took to solve this ODE is not the biggest of them all, even the ODE itself being the "scariest". This shows that not only the size of the ODE matters to the time of solving it. All these times were obtained on a machine with an Intel Celeron M processor 360 (1.40 GHz), 256 MB DDR SDRAM, running Windows XP Home.

ODE	Solution	Time
$\frac{dy}{dx} = \frac{y(1+x)}{x-xy-y^2+x^2}$	$\left(C + Ei\left(1, -\frac{x+y}{y}\right)\right)(y+x) + (y-y^2)e^{\frac{x+y}{y}} = 0$	$0.380s$
$\frac{dy}{dx} = -\frac{y^2(yax+yb+c)}{a^2x^2+2\,axb+b^2}$	$\int e^{-\frac{\left(ab+cy+a^2x\right)^2}{2ay^2(ax+b)^2}} \frac{(yax+yb+c)}{(ax+b)}dx + C\,y = 0$	$4.096s$
$\frac{dy}{dx} = \frac{y(1+x)}{x-y^2+x^2}$	$i\sqrt{\frac{\pi}{2}}\,erf\left(\frac{i\sqrt{2}x}{2y}\right) - e^{\frac{x^2}{2y^2}}y + C = 0$	$0.550s$
$\frac{dy}{dx} = \frac{-1+x+y+3\,y^2}{2(2\,x+y+xy+y^2-y^3)}$	$\sqrt{x+y^2}e^{-\frac{(-1+2\,y)^2}{4(x+y^2)}} - \frac{3}{2}\sqrt{\pi}\,erf\left(\frac{-1+2\,y}{2\sqrt{x+y^2}}\right) - C = 0$	$1.002s$
$\frac{dy}{dx} = -\frac{y^2\left(-2\,y+1-2\,x+x^2y\right)}{x^2(-2\,x+1-2\,y+y^2x)}$	$\int e^{\frac{(x+y)^2}{y^2x^2}} \frac{\left(-2\,y+1-2\,x+x^2y\right)}{x^2}dx + C = 0$	$1.562s$

Table 1: ODEs analyzed in the paper, their solutions and the corresponding time the program takes to solve them.

7 Conclusions

The Prelle-Singer approach is a semi-decision algorithm that can be used to solve analytically rational first order ordinary differential equations. Some implementations of it have been produced [18, 15]. In [15], we have introduced an extension to the PS-approach and we have presented an implementation of a package that extends the applicability of the PS-type methods in order to tackle some rational first order ordinary differential equations whose solutions contain a sub-class of (non-elementary) Liouvillian functions (therefore, outside the scope of the original PS-approach).

Here, we have implemented an extension presented on [1]. This extension now covers (in the same semi-decision way as the original PS approach does but for a wider class of differential equations) all rational first order ordinary differential equations with Liouvillian solutions. So, this enlarges the scope of the PS-type procedures previously implemented [18, 15] to allow it to tackle rational first order ODEs with Liouvillian solutions (see subsection 6.1). But, furthermore, it is worth mentioning that, as an additional bonus to this, we have enhanced the ODE solving capabilities in Maple, even for the cases where the ODE presents elementary solutions (a sub set of the Liouvillian ones). Since, in the present implementation, the integrating factor has a different general form: it is written with an exponential part and, in some cases, that translates to computational speed in the determination

of it since the integrating factor can be found with Darboux polynomials of much lower degree and that can be the difference between failure and success in solving the rational first order ordinary differential equations (see subsection 6.2).

In other words, apart from the theoretical extension that the present package now makes available for the user, thus making the analytical solving of a new class of rational first order ordinary differential equations possible, it also improves the Maple solving capabilities in the arena where the `dsolve` Maple command (even with some of its non-default enhancements turned on, e.g. the Lie package) and some previously made implementations of the PS-approach and its extensions [18, 15] already act.

As further development for this work, we are currently working on extensions to the package to tackle ODEs containing elementary functions (in the same fashion Shtokhamer [6] used for the usual PS procedure).

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